Section 4.4

The Shape of a Graph

(1) Concavity

(2) The Second Derivative Test

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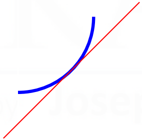
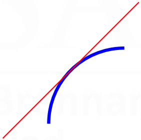
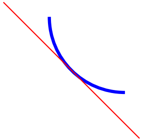
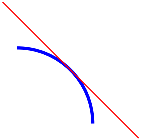
Concavity

Knowing the **direction** (increasing/decreasing) and **concavity** (up/down) of a curve tells us that it has one of four basic shapes.

	Concave up Curve above tangent line $f''(x)$ positive	**Concave down** Curve below tangent line $f''(x)$ negative
Increasing Positive slope $f'(x)$ positive		
Decreasing Negative slope $f'(x)$ negative		

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Concavity

Example 1: The function f records the temperature in degrees Celsius recorded t hours after the sun rises. At 3 hours after sunrise you are uncomfortably hot. How do you feel about each of the following scenarios?

a) $f'(3) = 2$ and $f''(3) = 4$ Worst situation! Getting hotter faster.

b) $f'(3) = -2$ and $f''(3) = 4$ Getting cooler but it will slow down.

c) $f'(3) = 2$ and $f''(3) = -4$ Getting hotter but it will slow down.

d) $f'(3) = -2$ and $f''(3) = -4$ Best situation! Getting cooler faster.

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Concavity and Extrema

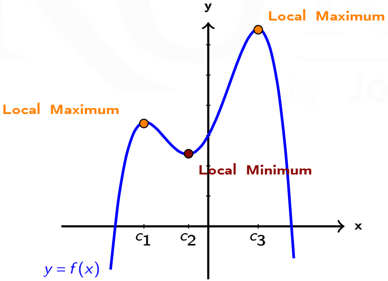
Second Derivative Test for Local Extrema

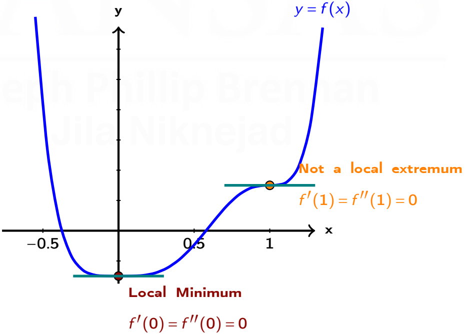
Suppose that f'' is continuous near c and $f'(c)=0$.

(I) If $f''(c)<0$, then $(c, f(c))$ is a local maximum.

(II) If $f''(c)>0$, then $(c, f(c))$ is a local minimum.

(III) If $f''(c)=0$, then we cannot draw any conclusion.





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Example 2: Use the 1st and 2nd Derivative Tests to find the local extrema of $f(x) = \dfrac{x^2}{x-1}$.

Solution: First find $f'(x)$ and $f''(x)$ (use the Quotient Rule):

$$f'(x) = \frac{x(x-2)}{(x-1)^2} \qquad f''(x) = \frac{2}{(x-1)^3}$$

Critical numbers: $0, 1, 2$. Note that $x = 1$ is not in the domain.

First Derivative Test:
Find the sign of $f'(x)$ in each interval between critical numbers.

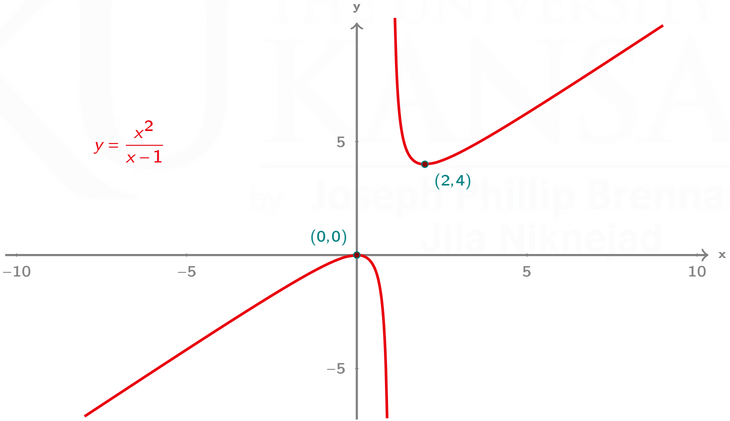
Conclusion: $(0,0)$ is a local max and $(2,4)$ is a local min.

Second Derivative Test:
Find the sign of $f''(x)$ at each critical number in the domain.

- $f''(0) = -2$: concave down, local max
- $f''(2) = 2$: concave up, local min

Example 2(continued): Use the 1st and 2nd Derivative Tests to find the local extrema of $f(x) = \dfrac{x^2}{x-1}$.

$f'(x) = \dfrac{x(x-2)}{(x-1)^2}$ $\qquad$ $f''(x) = \dfrac{2}{(x-1)^3}$



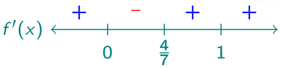
Example 3: Use the 1st and 2nd Derivative Tests to find the local extrema of $f(x) = x^4(x-1)^3$.

Solution: First find and simplify $f'(x)$ and $f''(x)$ (details omitted).

$$f'(x) = x^3(x-1)^2(7x-4) \qquad f''(x) = 6x^2(x-1)(7x^2-8x+2)$$

Critical numbers: $0, \frac{4}{7}, 1$.

First Derivative Test

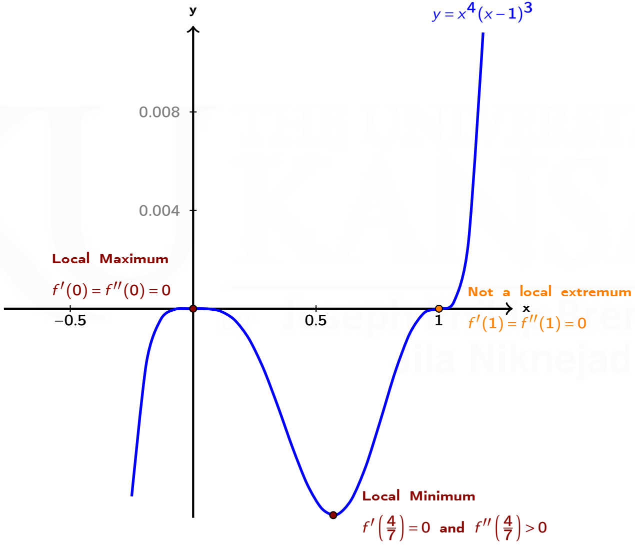


- $x = 0$: local max
- $x = \frac{4}{7}$: local min
- $x = 1$: neither

Second Derivative Test

- $f''(0) = 0$: inconclusive
- $f''(\frac{4}{7}) \approx 0.24 > 0$: local min
- $f''(1) = 0$: inconclusive

Example 3(continued): Use the 1st and 2nd Derivative Tests to find the local extrema of $f(x) = x^4(x-1)^3$.



First Derivative Test Versus Second Derivative Test

- Second derivative test does not require a table (number line) to find local extrema.
- If $f'(c) = f''(c) = 0$, then second derivative test is inconclusive so first derivative test can be used only. In this case, f may or may not have a local extremum at $x = c$.
- When you have the choice, use the second derivative test when finding f'' is not too complicated.
- First derivative test requires a table (number line) for values of f'.
- To find the shape of the graph, you may need to use both first derivative and second derivative. (Even if it is not necessary, it is recommended to do so to check your work.)

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